B.E.(M.D.U.)

First Semester Examination, 2008-09

Mathematics-1 (Math-1)

Note : Attempt **five** questions in all, selecting **two** questions from each part.

Part–A

Q. 1. (a) Test the convergence of the series :

$$1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+\frac{4^4}{5^5}+\ldots\ldots\infty$$

Ans. Leaving aside the first term

$$u_n=\frac{n^n}{(n+1)^{n+1}}=\frac{n^n}{n^{n+1}\left(1+\frac{1}{4}\right)^{n+1}}=\frac{1}{n\left(1+\frac{1}{4}\right)^{n+1}}$$

Take $\qquad u_n=\frac{1}{4}$

$$\underset{n\to\infty}{Lt}\frac{u_n}{v_n}=\underset{n\to\infty}{Lt}\frac{1}{\left(1+\frac{1}{4}\right)^{n+1}}=\underset{n\to\infty}{Lt}\frac{1}{\left(1+\frac{1}{4}\right)^{n}}\times\frac{1}{\left(1+\frac{1}{4}\right)}$$

$$=\frac{1}{e}\times\frac{1}{1}$$

which is finite & non-zero $\qquad \left[\because \underset{n\to\infty}{Lt}\left(1+\frac{1}{4}\right)^n=e\right]$

$\therefore\quad \Sigma u_n$ and Σv_n converge or diverge together since $\Sigma v_n=\Sigma\frac{1}{4}$ is of the form $\Sigma\frac{1}{4P}$ with $p=1$

$\therefore\quad \Sigma v_n$ is divergent $\Rightarrow \Sigma u_n$ is divergent.

Q. 1. (b) Examine the convergence or divergence of the following series :

$$\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+\ldots\infty(x>0)$$

Ans. $\qquad \frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\ldots\ldots$

Here $\qquad u_n=\frac{x^{2n-2}}{(n+1)\sqrt{4}} \qquad \therefore \qquad u_{n+1}=\frac{x^{2n}}{(n+2)\sqrt{n+1}}$

$$\frac{u_n}{u_{n+1}}=\frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{4}}\times\frac{1}{x^2}=\frac{1+\frac{2}{4}}{1+\frac{1}{4}}\sqrt{1+\frac{1}{4}}\times\frac{1}{x^2}$$

$\therefore \qquad \lim_{n \to \infty} \frac{u_n}{v_{n+1}} = \frac{1}{x^2}$

By D'Alembert's ratio test Σu_n-converges if $\frac{1}{x^2} > 1$

i.e., $x^2 < 1$ and diverges if $\frac{1}{x^2} > 1$ i.e., $x^2 > 1$

When $x^2 = 1, \qquad u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n^{3/2}\left(1+\frac{1}{4}\right)}$

Take $\qquad v_n = \frac{1}{n^{3/2}}, \qquad \lim_{n \to \infty} \frac{u_n}{v_n} = \frac{1}{\left(1+\frac{1}{4}\right)} = 1$

which is finite and non-zero. By comparison test Σu_n is convergent.

Here Σu_n is convergent if $x^2 \leq 1$ and divergent if $x^2 > 1$.

Q. 1. (c) For what values of x the following series is convergent :

$$x - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} - \frac{x^4}{\sqrt{4}} + \ldots \infty$$

Ans. The given series is

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^4}{\sqrt{n}}$$

Since $\qquad |u_n| = \frac{|x^n|}{\sqrt{n}} = \frac{|x|^n}{\sqrt{n}}$ and $(u_{n+1}) = \frac{|x|^{n+1}}{\sqrt{n+1}}$

$$\therefore \qquad \frac{|u_n|}{|u_{n+1}|} = \sqrt{\frac{n+1}{n}} \times \frac{1}{|x|} = \sqrt{1+\frac{1}{4}} \times \frac{1}{|x|}$$

$$\underset{n \to \infty}{Lt} \frac{|u_n|}{|u_{n+1}|} = \frac{1}{|x|}$$

$\therefore$ By ratio test, the series $\Sigma |u_n|$ is convergent if $\frac{1}{|x|} > 1$, i.e., if $|x| < 1$ i.e., if $-1 < x < 1$ and divergent if $\frac{1}{|x|} < 1$ i.e., $|x| > 1$ i.e., if $x > 1$ or $x < -1$

But ratio test fails when $|x| = 1 \qquad \therefore \qquad$ i.e., when $x = 1$ or $x = -1$

When $x = 1$ the series becomes $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots\ldots$

which is alternating series and is convergent when $x=-1$, the series become

$$-\left(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\ldots\ldots\right)=-\sum_{n=1}^{\infty} \frac{1}{\sqrt{4}}$$

which is divergent by p-test.

Hence the series is convergent if $-1<x\le 1$

Q. 2. (a) Expand sin x in powers of $\left(x-\frac{\pi}{2}\right)$. Hence find the value of sin 91° correct to 4 decimal places.

Ans.

$$f(x)=\sin x=f\left(\frac{\pi}{2}+x-\frac{\pi}{2}\right)$$

$$=f\left(\frac{\pi}{2}+h\right) \text{ where } h=x-\frac{\pi}{2}$$

$$=f\left(\frac{\pi}{2}\right)+hf'\left(\frac{\pi}{2}\right)+\frac{h^2}{2!}f''\left(\frac{\pi}{2}\right)+\frac{h^3}{3!}f'''\left(\frac{\pi}{2}\right)+\ldots \quad \ldots(1)$$

$$f(x)=\sin x, \qquad f\left(\frac{\pi}{2}\right)=1$$

$$f'(x)=\cos x \qquad f'\left(\frac{\pi}{2}\right)=0$$

$$f''(x)=-\sin x \qquad f''\left(\frac{\pi}{2}\right)=1$$

$$f'''(x)=-\cos x \qquad f'''\left(\frac{\pi}{2}\right)=0$$

$$f^{(vi)}(x)=\sin x \qquad f^{(iv)}\left(\frac{\pi}{2}\right)=1$$

$$\vdots \qquad \vdots$$

From equation (1)

$$f(x)=1+\left(x-\frac{\pi}{2}\right)(0)+\left(x-\frac{\pi}{2}\right)^2\times\frac{1}{2!}(-1)+\left(x-\frac{\pi}{3}\right)^3(0)+\left(x-\frac{\pi}{4}\right)^3\times\frac{1}{4!}$$

$$=1-\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}+\frac{\left(x-\frac{\pi}{4}\right)^4}{4}+\ldots$$

$$f(91)=f\left(\frac{\pi}{2}+1\right) \qquad h=1$$

$$\sin 91^\circ=0\cdot 9998$$

Q. 2. (b) Show that radius of curvature at the end of the major axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is equal to the semi-latus rectum.

Ans. Any point (x, y) on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } (a\cos\theta, b\sin\theta)$$

Parametric equation of the ellipse also

$$x=a\cos\theta, \qquad y=b\sin\theta$$

$$x'=-a\sin\theta, \qquad y'=b\cos\theta$$

$$x''=-a\cos\theta, \qquad y''=-b\sin\theta$$

$$\therefore \qquad S \text{ at } `\theta' = \frac{(x'^2+y'^2)^{3/2}}{x'y''-y'x''}=\frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}{ab\sin^2\theta+ab\cos^2\theta}$$

$$=\frac{(a^2\sin^2\theta+b^2\cos^2\theta)^{3/2}}{ab}$$

At the ends of major axis

$$\theta=0,\pi$$

$$S' \text{ at } \theta=0=\frac{[a(0)+b^2(1)]^{3/2}}{ab}=\frac{b^3}{ab}$$

$$=\frac{b^2}{a}$$

$=$ Semi latus rectum of ellipse

Q. 2. (c) Find the asymptotes of the curve

$$\boldsymbol{y^3-xy^2-x^2y+x^3+x^2-y^2=0}$$

Ans. $$y^3-xy^2-x^2y+x^3+x^2-y^2=0 \qquad \text{...(i)}$$

Since the coefficients of x^3 and y^3, the highest degree terms in x and y are constants, there are no asymptotes parallel to x axis and y axis. To find oblique asymptotes, putting $x=1$, $y=m$ in third, second & first degree in equation (i)

$$\phi_3(m)=m^3-m^2-m+1$$

$$\phi_2(m)=1-m^2$$

$$\phi_1(m)=0$$

The slope of asymptotes are roots of $\phi_3(m)=0$

$$m^3-m^2-m+1=0$$

$$m=1,\pm 1$$

$$m=-1$$

$$c = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{-(1-m^2)}{3m^2-2m-1}$$

$$m = -1$$

$$= \frac{-(1-1)}{3+2-1} = 0$$

$$y = mx + c$$

$$y = -1x \quad \Rightarrow \quad y + x$$

For repeated value of $m = 1$

$$\frac{c^2}{12}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$

$$\frac{c^2}{2}(6m-2) + c(-2m) + 0$$

$$\frac{c^2}{2}(4) - 2c = 0$$

$$2c^2 - 2c = 0$$

$$2c(c-1) = 0$$

$$c = 0, \qquad c = 1 \qquad y = mx + c$$

$$y = 1x + 0$$

$$y = 1x + 1$$

Q. 3. (a) If $r^2 = x^2 + y^2 + z^2$ and $V = r^m$, prove that $V_{xx} + V_{yy} + V_{zz} = m(m+1)r^{m-2}$.

Ans. $r^2 = x^2 + y^2 + z^2,\ V = r^m$

$$V_{xx} + V_{yy} + V_{zz} = m(m+1)r^{m-2}$$

Here $V = r^m = (x^2 + y^2 + z^2)^{m/2}$

$$V_x = \frac{m}{2}(x^2+y^2+z^2)^{\frac{m}{2}-1}2x$$

$$= mx(x^2+y^2+z^2)^{\frac{m-2}{2}}$$

$$V_{xx} = m\left[(x^2+y^2+z^2)^{\frac{m-2}{2}} + x\left(\frac{m-2}{2}\right)(x^2+y^2+z^2)^{\frac{m-2}{2}-1}2x\right]$$

$$= m\left[(x^2+y^2+z^2)^{\frac{m-2}{2}} + (m-2)x^2(x^2+y^2+z^2)^{\frac{m-4}{2}}\right]$$

$$\Rightarrow \qquad m\left[r^{m-2} + (m-2)x^2(r)^{m-4}\right] \qquad \ldots(1)$$

Similarly $V_{yy} = m[r^{m-2} + (m-2)y^2 r^{m-4}]$ ...(2)

$V_{zz} = m[r^{m-2} + (m-2)z^2 r^{m-4}]$ ...(3)

Equations (1) + (2) + (3) gives

$$V_{xx} + V_{yy} + V_{zz} = m[3r^{m-2} + (m-2)(r^{m-4})[x^2 + y^2 + z^2]]$$

$$= m[3r^{m-2} + (m-2)r^{m-4} r^2]$$

$$= m[3r^{m-2} + (m-2)r^{m-2}]$$

$\Rightarrow$ $$mr^{m-2}[3 + m - 2]$$

$$\boxed{V_{xr} + V_{yy} + V_{zz} \quad \Rightarrow \quad m(m+1)r^{m-2}}$$ **Hence Proved**

Q. 3. (b) If $u = \tan^{-1}\left(\dfrac{x^3 + y^3}{x - y}\right)$ prove that

$$\frac{x^2\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + \frac{y^2\partial^2 u}{\partial y^2} = 2\cos 3u \sin u$$

Ans. Here, Let $z = \tan u = \dfrac{x^3 + y^3}{x - y} = \dfrac{x^3(1 + y^3/x^3)}{x(1 - y/x)} = x^2 f(y/x)$

which is homogeneous of degree 2

By Euler's Theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

$\Rightarrow$ $$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$\Rightarrow$ $$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$\Rightarrow$ $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = \frac{2\sin u}{\cos u} \times \cos^2 u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\sin u\cos u = \sin 2u \quad ...(1)$$

Differentiating equation (i) wrt x

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\,\partial y} = 2\cos 2u\frac{\partial u}{\partial y} \quad ...(2)$$

Differentiating equation (i) wrt y

$$x\frac{\partial^2 y}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 2\cos 2u\frac{\partial u}{\partial y} \quad ...(3)$$

Apply equations $(2) \times x + (3) \times y$

$$x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$$

$$= 2 \cos 2u \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right] \quad ...(4)$$

Using equation (4)

$$\Rightarrow \quad x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + \sin 2u = 2 \cos 2u \sin 2u$$

$$\Rightarrow \quad x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \sin 2u \, (2 \cos 2u - 1)$$

$$\Rightarrow \quad \frac{x^2 \partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u$$

$$\boxed{x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \, \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2 \cos 3u \sin u}$$

Hence Proved

Q. 4. (a) Find the dimension of the rectangular box, open at the top, of maximum capacity whose surface is 432 sq. cm.

Ans. Let x, y, z be the length, breadth & height respectively. Let S be the given surface and V be the velocity or capacity.

$$S = xy + 2xz + 2yz = 432$$

$$f(x, y, z) \, V = xyz \quad ...(1)$$

$$\phi(x, y, z) = xy + 2yz + 2zx - 432 = 0 \quad ...(2)$$

Consider Lagrange's function

$$F(x, y, z) = xyz + \lambda \, (xy + 2yz + 2zx - 432)$$

For stationary value $dF = 0$

$$\Rightarrow \quad [y_2 + \lambda(y + 2z)] \, dx + [xz + \lambda(x + 2z)] \, dy + [xy + \lambda(2y + 2x)] \, dz = 0$$

$$yz + \lambda \, (y + 2z) = 0 \quad ...(3)$$

$$xz + \lambda(x + 2z) = 0 \quad ...(4)$$

$$xy + \lambda \, (2y + 2x) = 0 \quad ...(5)$$

Multiplying equation (3) by x and equation (4) by y and subtracting, we get

$$2z\lambda(x - y) = 0 \quad \Rightarrow \quad x = y$$

Multiplying equation (4) by y, equation (5) by z and subtracting, we get

$$x\lambda \, (y - 2z) = 0 \quad \Rightarrow \quad y = 2z$$

$$\therefore \quad x = y = 2z$$

From equation (2)

$$x \cdot x + x \cdot x + x \cdot x = 432$$

$$3x^2 = 432 \quad \Rightarrow \quad x^2 = 144$$

$$x = 12, \quad y = 12, \quad z = 6$$

Hence the dimensions of the box are 12 cm, 12 cm, 6 cm.

Q. 4. (b) Prove that

$$\int_0^{\pi} \frac{\log(1+\sin x \cos x)}{\cos x} dx = \pi\alpha$$

Ans. Here $\quad F(\alpha) = \int_0^{\pi} \frac{\log(1+\sin\alpha(\cos x))}{\cos x} dx = \pi\alpha$

$$F'(\alpha) = \frac{d}{d\alpha} \int_0^{\pi} \frac{\log(1+\sin\alpha \cos x)}{\cos x} dx$$

$$= \int_0^{\pi} \frac{\partial}{\partial\alpha} \frac{\log(1+\sin\alpha \cos x)}{\cos x} dx$$

$$= \int_0^{\pi} \frac{1}{(1+\sin\alpha \cos x)} \frac{(\cos\alpha \cos x)}{\cos x} dx$$

$$= \int_0^{\pi} \frac{\cos\alpha}{1+\sin\alpha \cos x} dx$$

Part–B

Q. 5. (a) Find the surface of the solid formed by revolving the cardioid $r = a(1+\cos\theta)$ about the initial line.

Ans. Since the upper and the lower halves of the cardioid generate the same volume, we consider the revolution of the upper half only for which r varies from 0 to $(1-\cos\theta)$ and θ varies from 0 to π

$\therefore$ Required volume of revolution

$$= 2\pi \int_0^{\pi} \int_0^{a(1-\cos\theta)} r^2 \sin\theta \, dr \, d\theta$$

$$= 2\pi \int_0^{\pi} \sin\theta \left(\frac{r^3}{3}\right)_0^{a(1-\cos\theta)} d\theta$$

$$= \frac{2\pi}{3} \int_0^{\pi} \sin\theta \, a^3 (1-\cos\theta)^3 \, d\theta$$

$$= \frac{2\pi a^3}{3} \int_0^{\pi} (1-\cos\theta)^3 \sin\theta \, d\theta$$

$$= \frac{2\pi a^3}{3}\left[\frac{(1-\cos\theta)^4}{4}\right]_0^{\pi} = \frac{\pi a^3}{6}(1-\cos\pi)^4$$

$$= \frac{\pi a^3}{6} 2^4 \qquad = \frac{8}{3}\pi a^3$$

Q. 5. (b) Find by double integration the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$.

Ans. Eliminating r between the equation of two curves, $\sin\theta = 1-\cos\theta$ or $\cos\theta + \sin\theta = 1$

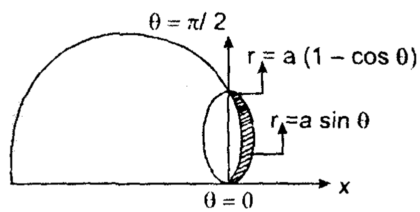
Equation $\quad 1+\sin 2\theta = 1 \quad$ or $\quad \sin 2\theta = 0$

$$2\theta = 0 \quad \text{or} \quad \pi$$

$$\theta = 0 \quad \text{or} \quad \frac{\pi}{2}$$

For the required area, r varies from $a(1-\cos\theta)$ to $a\sin\theta$ and θ varies from 0 to %

By Area



$$= \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left(\frac{r^2}{2}\right)_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} a^2\left[\sin^2\theta - (1-\cos\theta)^2\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left(\sin^2\theta - 1 - \cos^2\theta + 2\cos\theta\right) d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2}\left(-2\cos^2\theta + 2\cos\theta\right) d\theta$$

$$= a^2\left[-\frac{1}{2}\times\frac{\pi}{2} + 1\right] = a^2\left(1 - \frac{\pi}{4}\right)$$

Q. 6. (a) **Evaluate the following integral by changing to spherical co-ordinates :**

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$$

Ans. The region of integration is bounded by

$z=0,\qquad z=\sqrt{1-x^2-y^2}\qquad$ (i.e., $x^2+y^2+z^2=1$)

$x=0,\qquad x=1$

$y=0,\qquad y=\sqrt{1-x^2}\qquad$ (i.e., $x^2+y^2=1$)

which is volume of the sphere $x^2+y^2+z^2=1$ is the positive octant.

Changing to spherical polar co-ordinates by putting $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$

So that $\quad x^2+y^2+z^2=r^2$

For the volume of sphere $x^2+y^2+z^2=1$, in the positive octant, r varies from 0 to 1, θ varies from 0 to $\frac{\pi}{2}$ and ϕ varies from 0 to $\frac{\pi}{2}$

Replacing $dz\,dy\,dx$ by $r^2\sin\theta\,dr\,d\theta\,d\phi$, required

$$I=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{\sqrt{1-r^2}}$$

$$=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{1-(1-r^2)}{\sqrt{1-r^2}}\sin\theta\,dr\,d\theta\,d\phi$$

$$=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \left(\frac{1}{\sqrt{1-r^2}}-\sqrt{1-r^2}\right)\sin\theta\,dr\,d\theta\,d\phi$$

$$=\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\left[\sin^{-1}r-\left(\frac{r\sqrt{1-r^2}}{2}+\frac{1}{2}\sin^{-1}r\right)\right]_0^1 d\theta\,d\phi$$

$$=\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\left[\frac{\pi}{2}-\frac{1}{2}\frac{\pi}{2}\right]d\theta\,d\phi$$

$$=\int_0^{\pi/2} \frac{\pi}{4}(-\cos\theta)_0^{\pi/2}\,d\phi=\int_0^{\pi/2}\frac{\pi}{4}\,d\phi=\frac{\pi}{4}\times\frac{\pi}{2}$$

$$=\pi^2/8$$

Q. 6. (b) Prove that

$$\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$$

Ans. $$\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$$

Now $$\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} = \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta} \times \frac{\cos\theta}{2\sqrt{\sin\theta}}\,d\theta \qquad \left[\text{Putting } x^2 = \sin\theta,\ dx = \frac{\cos\theta\,d\theta}{2\sqrt{\sin\theta}}\right]$$

$$= \frac{1}{2}\int_0^{\pi/2} (\sin\theta)^{1/2}\,d\theta$$

$$= \frac{1}{4} B\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{[illegible]}\quad \Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right) [illegible]$$

$$\int_0^1 \frac{dx}{\sqrt{1+x^4}} = \int_0^{[illegible]} \frac{[illegible]}{2\sqrt{\tan\theta}\sec\theta} \qquad \left[\text{[illegible]ting } x^2 = \tan\theta,\ dx = \frac{\sec^2\theta\,d\theta}{2\sqrt{\tan\theta}}\right]$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi 4/2} \frac{d\theta}{\sqrt{\sin - 2\theta}} = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \sin^{-1/2}\phi\,d\phi$$

$$\left[\text{Putting } 2\theta = \phi, \qquad d\theta = \frac{1}{2}\,d\phi\right]$$

$$= \frac{1}{4\sqrt{2}} B\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4\sqrt{2}}\,\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(3/4)}$$

$$\therefore \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{1}{4\sqrt{2}}\left(\left(\Gamma\frac{1}{2}\right)\right)^2 = \frac{\pi}{4\sqrt{2}}$$ **Hence proved.**

Q. 7. (a) Find the directional derivative of $\phi = x^2\,yz + 4xz^2$ at the point $(1, -2, -1)$ in the direction of vector $2\hat{i} - \hat{j} - 2k$.

Ans. $$\phi = x^2yz + 4xz^2$$

$$\Delta\phi = \hat{i}\,\frac{\partial\phi}{\partial u} + \hat{j}\,\frac{\partial\phi}{\partial y} + \hat{k}\,\frac{\partial\phi}{\partial z}$$

$$=\hat{i}\,(2xyz+8xz)+\hat{j}\,(x^2z)+\hat{k}\,(x^2y+8xz)$$

$$=\hat{i}\,(2\,(1)(-2)(-1)+8\,(1)(-1))+\hat{j}\,(-1)+\hat{k}\,(-2+8\,(-1))$$

$$\Delta\phi\Big|_{(1,-2,-1)}=\hat{i}\,(4-8)\,\hat{j}-10\hat{k}$$

$$\Delta\phi=-4\hat{i}-\hat{j}-10\hat{k}$$

If $\hat{n}$ is a unit vector in the direction of $2\hat{i}-\hat{j}-2\hat{k}$

Then $$\hat{n}=\frac{2\hat{i}-\hat{j}-2\hat{k}}{\sqrt{4+1+4}}=\frac{1}{3}(2\hat{i}-\hat{j}-2\hat{k})$$

$\therefore$ Directional derivative of f in the direction of $\hat{n}$ is

$$\frac{1}{3}(-4\hat{i}-\hat{j}-10\hat{k})(2\hat{i}-\hat{j}-2\hat{k})$$

$$=\frac{1}{3}(-8+1+20)=\frac{13}{3}$$

Q. 7. (b) If $f=(x^2+y^2+z^2)^{-n}$, find div grad f and determine n if div grad $f=0$.

Ans. $$f=(x^2+y^2+z^2)^{-n}$$

Find div. grad f

$$\text{grad } f=\hat{i}\,\frac{\partial f}{\partial x}+\hat{j}\,\frac{\partial f}{\partial y}+\hat{k}\,\frac{\partial f}{\partial z}$$

$$=\hat{i}-n\,(x^2+y^2+z^2)^{n-1}\,2x+\hat{j}\,(-n)\,(x^2+y^2+z^2)\,2y$$
$$+\hat{k}\,(-n)\,(x^2+y^2+z^2)^{n-1}\,2z$$

$$=-\frac{2nx}{(x^2+y^2+z^2)^{n+1}}\,\hat{i}-\frac{2ny}{(x^2+y^2+z^2)^{n+1}}\,\hat{j}-\frac{2nz}{(x^2+y^2+z^2)^{n+1}}\,\hat{k}$$

$$\text{div (grad } f)=\frac{\partial}{\partial x}\left(\frac{-2nx}{(x^2+y^2+z^2)^{n+1}}\right)+\frac{\partial}{\partial y}\left(\frac{-2ny}{(x^2+y^2+z^2)^{n+1}}\right)$$
$$+\frac{\partial}{\partial z}\left(\frac{-2nz}{(x^2+y^2+z^2)^{n+1}}\right)$$

$\Rightarrow$ $$=\frac{\partial}{\partial x}\left(\frac{-2nx}{(x^2+y^2+z^2)^{n+1}}\right)=\frac{\partial}{\sqrt{x}}-2n\left[x\,(x^2+y^2+z^{2^{-(n+1)}}\right]$$

$$=-2n\left[x\,(-(n+1)\,(x^2+y^2+z^2)^{-n-2}\,2x+(x^2+y^2+z^2)^{(n+1)}\right]$$

$$=-2n\,[2x^2\,(-1)\,(n+1)\,(x^2-y^2+z^2)^{-(n+2)}$$
$$+(x^2+y^2+z^2)^{-(n+1)}] \qquad \ldots(1)$$

Similarly

$$\frac{\partial}{\partial y}\left(\frac{-2ny}{(x^2+y^2+z^2)^{n+1}}\right)=-2n[-2y^2\ (n+1)(x^2+y^2+z^2)^{-(n+2)}+(x^2+y^2+z^2)^{-(n+1)}] \quad ...(2)$$

$$\frac{\partial}{\partial z}\left(\frac{-2nz}{(x^2+y^2+z^2)^{n+1}}\right)=-2n[-2z^2\ (n+1)(x^2+y^2+z^2)^{-(n+2)}+(x^2+y^2+z^2)^{-(n+1)}] \quad ...(3)$$

(1) + (2) + (3)

$$\Rightarrow \quad -2n[-2(n+1)(x^2+y^2+z^2)^{-(n+2)}(x^2+y^2+z^2)+3(x^2+y^2+z^2)^{-(n+1)}]$$

$$\Rightarrow \quad -2n[-2(n+1)(r^2)^{-(n+2)}\,r^2+3(r^2)^{-(n+1)}]$$

$$\Rightarrow \quad -2n[-2(n+1)r^{-2n-4+2}+3r^{-2n-2}]$$

$$\Rightarrow \quad -2n[r^{-2n-2}(-2-2n+3)]$$

$$\Rightarrow \quad -2n\,r^{-2(n+1)}[1-2n]=0$$

When div (grad f) = 0

$$n=0, \qquad n=1/2$$

Q. 8. (a) Verify Green's theorem for

$$\int_C [3x-8y^2]dx+[4y-6xy]dy$$

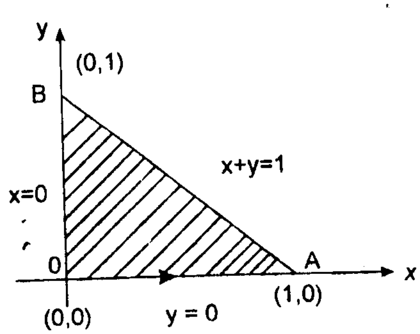
where C is the boundary of the region bounded by $x=0$, $y=0$ and $x+y=1$.

Ans. Here $\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$

Here $\quad M=3x^2-8y^2, \qquad N=4y-6xy$

$$\frac{\partial M}{\partial y}=-16y, \qquad \frac{\partial N}{\partial x}=-6y$$

$$\therefore \quad \frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}=10y$$



$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial N}{\partial y}\right) dx\, dy = \int_0^1 \int_0^{1-x} 10y\, dy\, dx$$

$$= \int_0^1 5(y^2)_0^{1-x}\, dx$$

$$= 5\int_0^1 (1-x)^2\, dx$$

$$= 5\left(\frac{(1-x)^3}{-3}\right)_0^1 = \frac{5}{3} \qquad \text{...(1)}$$

Along OA, $y = 0$ $\quad \therefore \quad$ $dy = 0$ and limits of x are from 0 to 1

$\therefore$ Line integral along OA $= \int_0^1 3x^2\, dx = [x^3]_0^1$

Along AB, $y = 1 - x$ $\quad \therefore \quad$ $dy = -dx$ and limits of x from 1 to 0

$\therefore$ Line integral along $AB = \int_1^0 [3x^2 - 8(1-x)^2]dx + [4(1-x) - 6x(1-x)(-dx)]$

$$= \int_1^0 (3x^2 - 8 + 16x - 8x^2 - 4 + 4x + 6x - 6x^2)\, dx$$

$$= \int_1^0 (-12 + 26x - 11x^2)\, dx$$

$$= (-12x + 13x^2 - \frac{11}{3}x^3)_1^0 = 8/3$$

Along BO, $x = 0$ $\quad \therefore \quad$ $dx = 0$ and limit of y are from 1 to 0

Line integral along $BO = \int_1^0 4y dy = -2$

Line integral along $C = 1 + \frac{8}{3} - 2 = \frac{5}{3}$

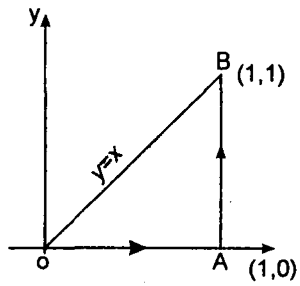
i.e., $\oint Mdx + Ndy = 5/3$ $\qquad$ ...(2)

The equality of equations (1) and (2) verifies Green's theorem in plane.

Q. 8. (b) Evaluate $\oint \vec{F} . \vec{dr}$ by Stoke's Theorem, where $\vec{F} = y^2\, \hat{i} + x^2\, \hat{j} - (x+z)\hat{k}$ and C is the boundary of triangle with vertices at (0, 0, 0), (1, 0, 0) and (1, 1, 0).

Ans. Since z-coordinates of each vertex of the triangle is zero, therefore, the triangle lies in the xy plane and $\hat{n} = \hat{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & x^2 & -(x+2) \end{vmatrix} = \hat{j} + 2(x-y)\hat{k}$$



$\therefore$
$$\text{curl } \vec{F} \cdot \hat{n} = [\hat{j} + 2(x-y)\hat{k}] \cdot \hat{k}$$
$$= 2(x-y)$$

The equation of the line OB is $y = x$

By Stoke's theorem $\oint_C \vec{F} \cdot \vec{dr} = \iint_S \text{curl } \vec{F} \cdot \hat{n}\, ds$

$$= \int_0^1 \int_0^x 2(x-y)\, dy\, dx$$

$$= \int_1^0 2\left(xy - \frac{y^2}{2}\right)_0^x dx = 2\int_0^1 \left(x^2 - \frac{x^2}{2}\right) dx$$

$$= \int_0^1 x^2 dx \quad = \frac{1}{3}$$